Nonexistence Results for Tight Block Designs

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Abstract

Recall that combinatorial $2s$-designs admit a classical lower bound $b \geq \binom{v}{s}$ on their number of blocks, and that a design meeting this bound is called tight. A long-standing result of Bannai is that there exist only finitely many nontrivial tight $2s$-designs for each fixed $s \geq 5$, although no concrete understanding of 'finitely many' is given. Here, we use the Smith Bound on approximate polynomial zeros to quantify this asymptotic nonexistence. Then, we outline and employ a computer search over the remaining parameter sets to establish (as expected) that there are in fact no such designs for $5 \leq s \leq 9$, although the same analysis could in principle be extended to larger s. Additionally, we obtain strong necessary conditions for existence in the difficult case $s = 4$.

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1 Introduction

Let $v \geq k \geq t$ be positive integers and λ be a nonnegative integer. A t-(v, k, λ) *design*, or simply a t-*design*, is a pair $(V, \mathcal{B})$ where V is a v-set and $\mathcal{B}$ is a collection of k-subsets of V such that any t-subset of V is contained in exactly λ elements of $\mathcal{B}$. The elements of V are *points* and the elements of $\mathcal{B}$ are *blocks*. Since t-designs are also i-designs for $i \leq t$, the parameter t is typically called the *strength*. The number of blocks is usually denoted b and an easy double-counting argument shows $b = \lambda\binom{v}{t}/\binom{k}{t}$.

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Suppose $(V, \mathcal{B})$ is a t-(v, k, λ) design. Generalizing Fisher's Inequality, Ray-Chaudhuri and Wilson [11] showed that if t is even, say $t = 2s$, and $v \geq k + s$, then $b \geq \binom{v}{s}$. If equality holds in this bound, we say $(V, \mathcal{B})$ is *tight.* The *trivial* tight $2s$-designs are those with $v = k + s$, where each of the $\binom{v}{k} = \binom{v}{s}$ k-subsets of V is a block. The case of odd strength is investigated in [7]; however, it is impossible for $(2s - 1)$-designs to be tight in the sense of having $\binom{v}{s-1}$ blocks.

Returning to even strength, the full set of parameters v and k for which a tight $2s$-design exists has only been determined for $s = 2, 3$. Note that, when $s = 1$, tight 2-designs have $b = v$ and are the 'symmetric' designs; see [6, 9] for surveys of this rich (yet very challenging) topic. In 1975, Ito [8] published a proof that the only nontrivial tight 4-designs are the Witt 4-(23,7,1) design and its complementary 4-(23,16,52) design, but his proof was found to be incorrect. A few years later, Enomoto, Ito, and Noda [5] proved the weaker result that there are finitely many nontrivial tight 4-designs, though still believing Ito's initial claim to be true. Finally, in 1978, Bremner [2] successfully settled $s = 2$ by reaffirming Ito's result. Peterson [10] proved in 1976 that there exist no nontrivial tight 6-designs.

Bannai [1] proved that there exist only finitely many nontrivial tight $2s$-designs for each $s \geq 5$. The case $s = 4$ is quite open, and the 'finitely many' for $s \geq 5$ is not explicit and potentially grows with s. However, it is probably the case that there are no unknown tight $2s$-designs for $s \geq 2$.

Central to these negative results is the following strong condition, discovered first by Ray Chaudhuri and Wilson [11], and also implicitly by Delsarte [3].

Proposition 1.1. ([3, 11]) *If there exists a tight $2s$-(v, k, λ) design, then the zeros of the following degree s polynomial $\Psi_s(x)$ are the intersection numbers of the design, and hence they must all be nonnegative integers:*

$$\Psi_s(x) = \sum_{i=0}^{s} (-1)^{s-i} \frac{\binom{v-s}{i}\binom{k-i}{s-i}\binom{k-1-i}{s-i}}{\binom{s}{i}} \binom{x}{i}. \tag{1.1}$$

The polynomials Ψ_s are variants of the Hahn polynomials, [4].

Since a $2s$-design with $v \geq k + s$ induces at least s intersection numbers [11], it follows that the zeros of Ψ_s must additionally be distinct integers for tight designs. Note also that Ψ_s has no dependence on λ; indeed, for tight designs $\lambda = \binom{v}{s}\binom{k}{2s}\binom{v}{2s}^{-1}$ and is therefore uniquely determined by v and k.

Analogously, the Lloyd polynomials $L_e(x)$ are important for the characterization of perfect e-error-correcting codes; see [15]. It is interesting that this characterization of perfect codes was completed long ago, while the open problems mentioned before Proposition 1.1 remain for tight designs. Our goal here is to revive the interest in tight designs and take a modest step toward the full characterization of their parameters.

The outline is as follows. In Section 2, we review the work of Bannai in [1] on the asymptotic structure of the zeros of Ψ_s. Extending this, we obtain some exact bounds relevant to this analysis. Section 3 summarizes the techniques for exhausting small cases $s \geq 5$, and Section 4 is devoted to a partial analysis of

the case $s = 4$. An appendix of tables following the main text will prove useful to the interested reader.

2 Bannai's analysis and the Smith bound

2.1 Notation

Assuming a tight design, let x_i, for $i = -\lfloor \frac{s}{2} \rfloor, \ldots, (0), \ldots, \lfloor \frac{s}{2} \rfloor$, denote the zeros of Ψ_s listed in increasing order. For example, the zeros of Ψ_4 and Ψ_5 are denoted $x_{-2} < x_{-1} < x_1 < x_2$ and $x_{-2} < x_{-1} < x_0 < x_1 < x_2$, respectively.

An important parameter is the arithmetic mean of the zeros of $\Psi_s(x)$, which we denote by $\overline{\alpha}$. From the coefficient of x^{s-1}, we have

$$\overline{\alpha} = \frac{(k-s+1)(k-s)}{v-2s+1} + \frac{s-1}{2}. \tag{2.1}$$

Now define, as in [1],

$$\alpha = \frac{(k-s+1)(k-s)}{v-2s+1},$$

so that $\overline{\alpha} = \alpha + (s-1)/2$. Also, following Bannai's notation, let us redefine the parameter t as

$$t = \frac{v-2s+1}{k-s+1}.$$

Note $t = 2$ implies $v = 2k+1$. Moreover, if $v < 2k$, we may complement blocks, replacing k with $v-k$ and obtain $v > 2k$. This is discussed further in Section 2.2.

Finally, put $\beta = \left(1 - \frac{1}{t}\right)\sqrt{\alpha}$. In terms of v and k,

$$\beta = \frac{(v-k-s)\sqrt{(k-s+1)(k-s)}}{(v-2s+1)^{3/2}}.$$

So, in particular, $\beta = 0$ if and only if $v = k+s$. In some sense β can be seen as measuring the 'nontriviality' of a (tight) $2s$-design. Note also that

$$k = t^3(t-1)^{-2}\beta^2 + s, \text{ and} \tag{2.2}$$

$$v = t^4(t-1)^{-2}\beta^2 + t + 2s - 1. \tag{2.3}$$

Bannai's proof of the existence of only finitely many nontrivial tight $2s$-designs, $s \geq 5$, is divided into cases according to this parameter β. In particular, he proves

- for any β_0, there are only finitely many tight $2s$-designs with $\beta \leq \beta_0$; and
- there exists β_0 (depending only on s), such that there are no nontrivial tight $2s$-designs with $\beta > \beta_0$.

Here, our main goal is to compute such a β_0 explicitly for $5 \leq s \leq 9$ and, by searching across all pairs (v, k) for which $\beta \leq \beta_0$, show that there are in fact zero nontrivial tight $2s$-designs for these s.

2.2 Symmetry with respect to the parameter t

In the analytic work which follows, it is helpful to obtain a lower bound on t. As discussed above, we may complement blocks to assume $v \geq 2k$. The following was mentioned but not fully proven in [1].

Lemma 2.1. *Let $s \geq 1$. There are no tight $2s$-designs with $v = 2k$.*

Proof: Suppose $v = 2k$. Then from (2.1), $s\overline{\alpha} - \binom{s}{2} = s\alpha = \frac{s(k-s+1)(k-s)}{2(k-s)+1}$. Without too much effort, it can be seen that the least residue of $s(k-s+1)(k-s) \pmod{2(k-s)+1}$, denoted here by $r_{k,s}$, satisfies

$$r_{k,s} = \begin{cases} 2(k-s) - \frac{s-4}{4} & \text{if } s \equiv 0 \pmod 4; \\ k - s - \frac{s-2}{4} & \text{if } s \equiv 2 \pmod 4; \\ \frac{k-s}{2} - \frac{s-1}{4} & \text{if } s \equiv 1 \pmod 4 \text{ and } k \text{ is odd} \\ & \quad \text{or } s \equiv 3 \pmod 4 \text{ and } k \text{ is even}; \\ \frac{3}{2}(k-s) - \frac{s-3}{4} & \text{if } s \equiv 3 \pmod 4 \text{ and } k \text{ is odd} \\ & \quad \text{or } s \equiv 1 \pmod 4 \text{ and } k \text{ is even}. \end{cases}$$

Since $k - s \geq s$, it follows that in all cases $r_{k,s}$ is an integer lying strictly between 0 and $2(k-s)+1$, so $s\overline{\alpha} - \binom{s}{2}$ is not an integer. But the integrality of $s\overline{\alpha}$ is necessary for the existence of a tight design since it is the sum of the zeros of $\Psi_s(x)$; therefore there are no tight $2s$-designs with $v = 2k$. □

Now, we are able to justify assuming that $t \geq 2$ for nonexistence of tight designs.

Proposition 2.2. *Let $s \geq 1$. If there exists a nontrivial tight $2s$-design with $t < 2$, then there also exists a nontrivial tight $2s$-design with $t \geq 2$.*

Proof: Suppose $\mathcal{D}$ is a nontrivial tight $2s$-(v, k, λ) design with $t < 2$. This means $k \leq v \leq 2k-1$ because $v \neq 2k$ by Lemma 2.1, and so the complementary $2s$-$(v, v-k, \lambda')$ design of $\mathcal{D}$ is a nontrivial tight $2s$-design with $t \geq 2$. □

Incidentally, Bannai and Peterson ruled out the case $t = 2$, observing that it yields symmetric zeros of Ψ_s about their mean $\overline{\alpha}$. This is a key observation.

Proposition 2.3. ([1, 10]) *There does not exist any tight $2s$-design with $v = 2k+1$.*

2.3 Hermite polynomials

Let $H_s(x)$ denote the normalized Hermite polynomial of degree s defined recursively by $H_0(x) = 1$, $H_1(x) = x$, and for $s \geq 2$,

$$H_s(x) = xH_{s-1}(x) - (s-1)H_{s-2}(x).$$

Furthermore, let ξ_i, $i = -\left\lfloor \frac{s}{2} \right\rfloor, \ldots, (0), \ldots, \left\lfloor \frac{s}{2} \right\rfloor$, denote the zeros of $H_s(x)$ listed in increasing order. It is easily seen that $\xi_{-i} = -\xi_i$ for each i. See Appendix A

for a table of zeros of $H_s(x)$, $1 \le s \le 10$. In particular, for the analytical work in Section 3, we will make use of the following known estimates.

Proposition 2.4.

(i) *If s is odd and ≥ 5, then $\xi_1^2 < \sqrt{3}$.*

(ii) *If s is even and ≥ 8, then $\xi_2^2 - \xi_1^2 < \sqrt{3}$.*

(iii) *If $s = 6$, then $1.0 < \frac{\xi_2^2 - \xi_1^2}{3} < 1.1$, $3.5 < \frac{\xi_3^2 - \xi_1^2}{3} < 3.6$, and $3.34634 < \frac{\xi_3^2 - \xi_1^2}{\xi_2^2 - \xi_1^2} < 3.34635$.*

Proof: Items (i) and (ii) are referenced in Bannai's Proposition 13 and proven on page 126 of [14]. Item (iii) can be verified numerically. See Appendix A. (Note that Bannai's Proposition 13 (iii) actually contains an error). □

A useful identity is

$$H_s'(x) = sH_{s-1}(x). \tag{2.4}$$

For later reference we define, again as in [1],

$$\lambda_i = \lambda_i(t) = \left(1 - \tfrac{2}{t}\right)^2 \left(\tfrac{\xi_i^2}{6} - \tfrac{s-1}{6}\right). \tag{2.5}$$

Informally, Proposition 16 in [1] states that as $\beta \to \infty$, the zeros x_i of $\Psi_s(x)$ approach $\overline{\alpha} + \beta\xi_i + \lambda_i$. That is, when suitably normalized, Ψ_s behaves like H_s for large β and fixed t.

2.4 The Smith bound

We now state a useful result for explicitly finding β_0. Sometimes known as the Smith bound, it is a consequence of the Gershgorin circle theorem.

Theorem 2.5. ([13]) *Let $P(z)$ be a monic polynomial of degree n and let $\xi_1, \ldots, \xi_n$ be distinct points approximating the zeros of $P(z)$. Define the circles*

$$\Gamma_i = \left\{ z : |z - \xi_i| \le \frac{n|P(\xi_i)|}{|Q'(\xi_i)|} \right\},$$

where $Q(z)$ is the monic polynomial of degree n with zeros $\xi_1, \ldots, \xi_n$. Then the union of the circular regions Γ_i contains all the zeros of $P(z)$, and any connected component consisting of just k circles Γ_i contains exactly k zeros of $P(z)$.

Let $s \ge 1$. For each $i \in \{-\lfloor \frac{s}{2} \rfloor, \ldots, (0), \ldots, \lfloor \frac{s}{2} \rfloor\}$, define the monic degree s polynomial (in z)

$$G_s^{(i)}(z) = \frac{s!}{\beta^s \binom{v-s}{s}} \Psi_s(\overline{\alpha} + \beta z + \lambda_i), \tag{2.6}$$

and put $z_i = (x_i - \overline{\alpha} - \lambda_i)/\beta$, the zero of $G_s^{(i)}(z)$ corresponding to x_i.

We will see from Propositions 2.6 and 2.7 that the z_i are well-approximated by the ξ_i as $\beta \to \infty$, independently of t.

Proposition 2.6. *Let $s \geq 1$. Then*

$$|z_i - \xi_i| \leq \frac{|G_s^{(i)}(\xi_i)|}{|H_{s-1}(\xi_i)|}.$$

Proof: Simply apply Theorem 2.5 to the polynomial $G_s^{(i)}(z)$, letting $Q(z) = H_s(z)$, to get

$$|z_i - \xi_i| \leq \frac{s|G_s^{(i)}(\xi_i)|}{|H_s'(\xi_i)|}.$$

The result now follows from (2.4). □

2.5 Bounding G_s in terms of β

In the next proposition, it is helpful to think of the $G_s^{(i)}(\xi_i)$ as functions of β and t.

Proposition 2.7. *Let $s \geq 2$. For each $i \in \{-\lfloor \frac{s}{2} \rfloor, \ldots, (0), \ldots, \lfloor \frac{s}{2} \rfloor\}$, there exist constants B_i, C_i such that whenever $\beta > B_i$,*

$$|G_s^{(i)}(\xi_i)| < \frac{C_i}{\beta^2}$$

for all $t \geq 2$.

The necessary ingredients for this result were proved in [1], although the bound was not directly stated in this form. Therefore, we omit the proof and instead focus on how to (carefully) obtain B_i and C_i for small s using some basic computer algebra.

Algorithm 2.8. For fixed s and i, we may obtain constants B_i and C_i in Proposition 2.7 by the following procedure.

1. Using (1.1), substitute (2.1), (2.2), (2.3) and (2.5) into (2.6). To defer floating-point precision issues, we first replace ξ_i in (2.5) by a symbolic parameter r.

2. This results in an expression for $G_s^{(i)}(r)$ as a rational function of β, say

$$G_s^{(i)}(r)(\beta, t) = \frac{p(r, \beta, t)}{q(\beta, t)}.$$

   Here, the denominator is

$$q(\beta, t) = \beta^s \binom{v-s}{s} = \beta^s \binom{t^4(t-1)^{-2}\beta^2 + t + s - 1}{s}. \tag{2.7}$$

3. Observe that q is positive for $\beta > 0$ and $t \geq 2$, and that a lower bound on q is

$$\tilde{q}(\beta, t) = \tfrac{1}{s!}\beta^{3s} t^{4s} (t-1)^{-2s}.$$

   This is obtained by replacing each factor in the falling factorial of (2.7) by $t^4(t-1)^{-2}\beta^2$.

4. The numerator $p(r, \beta, t)$ is, for general r, a polynomial of degree $3s$ in β. However, for $r = \xi_i$, Proposition 2.7 shows the two top coefficients, namely of β^{3s} and β^{3s-1}, vanish. Again, to maintain symbolic algebra, we artifically replace these coefficients by zero and call this polynomial $\tilde{p}(r, \beta, t)$.

5. We have
$$\beta^2 G_s^{(i)}(r)(\beta, t) \leq \frac{\beta^2 \tilde{p}(r, \beta, t)}{\tilde{q}(\beta, t)}.$$
Note that for $r = \xi_i$, the right hand side is a polynomial in β^{-1}.

6. Consider the coefficient $\kappa_j(r, t)$ of β^{3s-j} in $\beta^2 \tilde{p}(r, \beta, t)$. With $r = \xi_i$, compute (or upper-bound) the maxima
$$M_j = \sup_{t \geq 2} \frac{|\kappa_j(\xi_i, t)|}{\frac{1}{s!}(t-1)^{-2s} t^{4s}}.$$
Then, estimating term-by-term,
$$|\beta^2 G_s^{(i)}(\xi_i)(\beta, t)| \leq M_0 + M_1 \beta^{-1} + M_2 \beta^{-2} + \dots$$
for all $t \geq 2$.

7. Construct B_i, C_i so that $\beta > B_i$ implies $M_0 + M_1\beta^{-1} + M_2\beta^{-2} + \dots \leq C_i$. Note that with sufficiently large B_i and a safe choice of C_i, it suffices to estimate the first few coefficients M_j.

We should remark that for small s, Algorithm 2.8 – even the calculation of all $3s - 1$ coefficient maxima M_j – is essentially instantaneous on today's personal computers. Moreover, deferring the use of floating-point arithmetic to step 5 – when t is eliminated – makes our subsequent use of floating-point numbers M_j quite mild. Indeed, there is virtually no loss in taking M_j as (integer) ceilings of the suprema, so that estimating for C_i can be performed in $\mathbb{Q}$.

See Appendix B for the results of this calculation for each $5 \leq s \leq 9$ and all relevant indices i.

2.6 Bounding the zeros

We are now ready for our main result of this section. This is in Bannai's paper [1], but with no attempt to control β.

Proposition 2.9. *Fix a positive integer s and $i \in \{-\lfloor \frac{s}{2} \rfloor, \dots, (0), \dots, \lfloor \frac{s}{2} \rfloor\}$. Put $y_i = x_i - \overline{\alpha} - \beta\xi_i$, where recall x_i and ξ_i are corresponding roots of Ψ_s and H_s, respectively. Let $\epsilon > 0$ and define*
$$\widehat{\beta}(i, \epsilon) = \max\left\{B_i, \frac{C_i}{\epsilon D_i}\right\},$$
where $D_i = |H_{s-1}(\xi_i)|$. Then for all $\beta > \widehat{\beta}$ and all $t \geq 2$,
$$|y_i - \lambda_i| < \epsilon.$$

Proof: Observe that $|y_i - \lambda_i| = \beta|z_i - \xi_i|$, since

$$x_i = \overline{\alpha} + \beta\xi_i + y_i = \overline{\alpha} + \beta z_i + \lambda_i.$$

The estimate now follows easily from Propositions 2.6 and 2.7. $\square$

3 The case $s \geq 5$

3.1 Estimates for large β

The goal here is to provide formulas for the smallest β_0 possible (see the end of Section 2.1) using the B_i and C_i constructed in Algorithm 2.8. This task is simplified under the conditions that B_i is independent of i and $C_i = C_{-i}$. There is no loss of generality in assuming this because we can simply take $\overline{B}$ to be the maximum of the B_i and $\overline{C_i} = \max\{C_i, C_{-i}\}$, and then redefine each $B_i = \overline{B}$ and $C_i = C_{-i} = \overline{C_i}$. In fact, this is not necessary for our explicit constructions because the constants in Appendix B satisfy the above conditions.

Again, for convenience, we denote $|H_{s-1}(\xi_i)|$ by D_i in the following proofs.

Proposition 3.1. *Let $s \geq 5$ be odd.*

(i) *There exists β_1 such that, whenever $\beta > \beta_1$,*

$$|y_1 + y_{-1} - 2y_0| < 1.$$

(ii) *There exists β_2 such that, whenever $\beta > \beta_2$,*

$$|y_i + y_{-i} - y_{i-1} - y_{-(i-1)}| < 1 + \frac{\xi_i^2 - \xi_{i-1}^2}{\xi_{i-1}^2 - \xi_{i-2}^2}|y_{i-1} + y_{-(i-1)} - y_{i-2} - y_{-(i-2)}|$$

for $2 \leq i \leq \left\lfloor \frac{s}{2} \right\rfloor$.

(iii) *There exists $\beta_0(s)$ such that, whenever $\beta > \beta_0$ and $y_i + y_{-i} - y_{i-1} - y_{-(i-1)}$ is an integer for $1 \leq i \leq \left\lfloor \frac{s}{2} \right\rfloor$, it is necessarily the case that $y_i + y_{-i} - y_{i-1} - y_{-(i-1)} = 0$ for each i.*

Proof:

(i) Observe that since $t \geq 2$,

$$0 \leq 2(\lambda_1 - \lambda_0) = (1 - \tfrac{2}{t})^2 \tfrac{\xi_1^2}{3} < \tfrac{\xi_1^2}{3}. \tag{3.1}$$

Define

$$\epsilon_0 = \frac{1}{2}\left(1 - \frac{\xi_1^2}{3}\right)\left(1 + \frac{C_1 D_0}{C_0 D_1}\right)^{-1} \quad \text{and} \quad \beta_1 = \widehat{\beta}(0, \epsilon_0).$$

If $\epsilon_1 = \epsilon_0 \frac{C_1 D_0}{C_0 D_1}$, then

$$\widehat{\beta}(1, \epsilon_1) = \beta_1 \quad \text{and} \quad 2\epsilon_0 + 2\epsilon_1 = 1 - \frac{\xi_1^2}{3}.$$

Hence for $\beta > \beta_1$,

$$\begin{aligned} |y_1 + y_{-1} - 2y_0 - 2(\lambda_1 - \lambda_0)| &\leq |y_1 - \lambda_1| + |y_{-1} - \lambda_{-1}| + 2|y_0 - \lambda_0|, \\ &< 2\epsilon_0 + 2\epsilon_1 = 1 - \frac{\xi_1^2}{3}, \end{aligned}$$

By (3.1),

$$-\left(1 - \tfrac{\xi_1^2}{3}\right) < y_1 + y_{-1} - 2y_0 < 1$$

and the claim follows.

(ii) For $2 \leq i \leq \left\lfloor \frac{s}{2} \right\rfloor$, let

$$a_i = \frac{\xi_i^2 - \xi_{i-1}^2}{\xi_{i-1}^2 - \xi_{i-2}^2} \quad \text{and} \quad \epsilon_i = \frac{1}{2}\left(1 + (1 + a_i)\frac{C_{i-1}D_i}{C_i D_{i-1}} + a_i \frac{C_{i-2}D_i}{C_i D_{i-2}}\right)^{-1}.$$

Note if $2 \leq i \leq \left\lfloor \frac{s}{2} \right\rfloor$, then

$$(\lambda_i - \lambda_{i-1}) = (\lambda_{i-1} - \lambda_{i-2})a_i. \tag{3.2}$$

Define $\beta_2 = \max\left\{\widehat{\beta}(i, \epsilon_i) : 2 \leq i \leq \left\lfloor \frac{s}{2} \right\rfloor\right\}$. For $\beta > \beta_2$ and working as in (i),

$$|y_i + y_{-i} - y_{i-1} - y_{-(i-1)} - 2(\lambda_i - \lambda_{i-1})| < 2\epsilon_i \left(1 + \tfrac{C_{i-1}D_i}{C_i D_{i-1}}\right). \tag{3.3}$$

Using (3.2) and (3.3) again with $i - 1$ replacing i,

$$\begin{aligned} &|y_i + y_{-i} - y_{i-1} - y_{-(i-1)}| \\ &\quad < 2\epsilon_i \left(1 + \tfrac{C_{i-1}D_i}{C_i D_{i-1}}\right) + 2\epsilon_i a_i \left(\tfrac{C_{i-1}D_i}{C_i D_{i-1}} + \tfrac{C_{i-2}D_i}{C_i D_{i-2}}\right) \\ &\qquad + a_i |y_{i-1} + y_{-(i-1)} - y_{i-2} - y_{-(i-2)}| \\ &\quad = 1 + a_i |y_{i-1} + y_{-(i-1)} - y_{i-2} - y_{-(i-2)}|, \end{aligned}$$

as required.

(iii) Set $\beta_0(s) = \max\{\beta_1, \beta_2\}$ and assume that $\beta > \beta_0(s)$ and $y_i + y_{-i} - y_{i-1} - y_{-(i-1)}$ is an integer for $1 \leq i \leq \left\lfloor \frac{s}{2} \right\rfloor$. By (i), $y_1 + y_{-1} - 2y_0 = 0$ since it is an integer whose absolute value is less than 1. Assume that $y_{i-1} + y_{-(i-1)} - y_{i-2} - y_{-(i-2)} = 0$ for some $2 \leq i \leq \left\lfloor \frac{s}{2} \right\rfloor$. Then (ii) gives that $|y_i + y_{-i} - y_{i-1} - y_{-(i-1)}|$ is also less than one and hence equal to 0 since it is an integer, so by induction $y_i + y_{-i} - y_{i-1} - y_{-(i-1)} = 0$ for $1 \leq i \leq \left\lfloor \frac{s}{2} \right\rfloor$, and so the proof is complete. □

Proposition 3.2. *Let $s \geq 8$ be even.*

(i) *There exists β_1 such that, whenever $\beta > \beta_1$,*

$$|y_2 + y_{-2} - y_1 - y_{-1}| < 1.$$

(ii) *There exists* β_2 *such that, whenever* $\beta > \beta_2$,

$$|y_i + y_{-i} - y_{i-1} - y_{-(i-1)}| < 1 + \frac{\xi_i^2 - \xi_{i-1}^2}{\xi_{i-1}^2 - \xi_{i-2}^2} |y_{i-1} + y_{-(i-1)} - y_{i-2} - y_{-(i-2)}|$$

for $3 \le i \le \lfloor \frac{s}{2} \rfloor$.

(iii) *There exists* $\beta_0(s)$ *such that, whenever* $\beta > \beta_0(s)$ *and* $y_i + y_{-i} - y_{i-1} - y_{-(i-1)}$ *is an integer for* $2 \le i \le \lfloor \frac{s}{2} \rfloor$, *it is necessarily the case that* $y_i + y_{-i} - y_{i-1} - y_{-(i-1)} = 0$ *for each* i.

Proof:

(i) Since $t \ge 2$,

$$0 \le 2(\lambda_2 - \lambda_1) = (1 - \tfrac{2}{t})^2 \tfrac{\xi_2^2 - \xi_1^2}{3} < \tfrac{\xi_2^2 - \xi_1^2}{3}. \tag{3.4}$$

Define

$$\epsilon_1 = \frac{1}{2}\left(1 - \frac{\xi_2^2 - \xi_1^2}{3}\right)\left(1 + \frac{C_2 D_1}{C_1 D_2}\right)^{-1} \quad \text{and} \quad \beta_1 = \widehat{\beta}(1, \epsilon_1).$$

If $\epsilon_2 = \epsilon_1 \frac{C_2 D_1}{C_1 D_2}$, then

$$\widehat{\beta}(2, \epsilon_2) = \beta_1 \quad \text{and} \quad 2\epsilon_1 + 2\epsilon_2 = 1 - \frac{\xi_2^2 - \xi_1^2}{3}.$$

Hence for $\beta > \beta_1$,

$$|y_2 + y_{-2} - y_1 - y_{-1} - 2(\lambda_2 - \lambda_1)| < 2\epsilon_1 + 2\epsilon_2 = 1 - \frac{\xi_2^2 - \xi_1^2}{3}.$$

By (3.4),

$$-\left(1 - \tfrac{\xi_2^2 - \xi_1^2}{3}\right) < y_2 + y_{-2} - y_1 - y_{-1} < 1$$

and the claim follows.

(ii) Define ϵ_i and β_2 in as in the proof of Proposition 3.1 (ii), but omit $i = 2$.

(iii) Imitate the proof of Proposition 3.1 (iii). □

In the case $s = 6$, $\frac{\xi_2^2 - \xi_1^2}{3} > 1$. Hence it is impossible to choose a β_1 to guarantee that $y_2 + y_{-2} - y_1 - y_{-1} = 0$ whenever it is an integer and $\beta > \beta_1$.

Proposition 3.3. *Let* $s = 6$. *There exists* $\beta_0(6)$ *such that, whenever* $\beta > \beta_0(6)$ *and* $(y_2 + y_{-2} - y_1 - y_{-1})$, $(y_3 + y_{-3} - y_1 - y_{-1})$ *are both integers, it is necessarily the case that* $y_2 + y_{-2} - y_1 - y_{-1} = y_3 + y_{-3} - y_1 - y_{-1} = 0$.

Proof: Observe

$$0 \le 2(\lambda_2 - \lambda_1) < \tfrac{\xi_2^2 - \xi_1^2}{3} < 1.1 \quad \text{and} \quad 0 \le 2(\lambda_3 - \lambda_1) < \tfrac{\xi_3^2 - \xi_1^2}{3} < 3.6$$

by Proposition 2.4 (iii). Let $a = \frac{\xi_3^2 - \xi_1^2}{\xi_2^2 - \xi_1^2}$ and define

$$\epsilon_1 = \frac{1}{2}(a-3)\left(1 + a + a\frac{C_2 D_1}{C_1 D_2} + \frac{C_3 D_1}{C_1 D_3}\right)^{-1} \quad \text{and} \quad \beta_0(6) = \widehat{\beta}(1, \epsilon_1).$$

Then, with

$$\epsilon_2 = 2\epsilon_1\left(1 + \frac{C_2 D_1}{C_1 D_2}\right) \quad \text{and} \quad \epsilon_3 = 2\epsilon_1\left(1 + \frac{C_3 D_1}{C_1 D_3}\right),$$

we have

$$\begin{gathered} \epsilon_2 a + \epsilon_3 = a - 3, \\ 0 < \epsilon_2 < (a-3)/a \approx 0.10350, \quad \text{and} \\ 0 < \epsilon_3 < a - 3 \approx 0.34635. \end{gathered}$$

Assume $\beta > \beta_0(6)$. Then $|y_2 + y_{-2} - y_1 - y_{-1} - 2(\lambda_2 - \lambda_1)| < \epsilon_2$ implies

$$y_2 + y_{-2} - y_1 - y_{-1} \in \{0, 1\}.$$

Likewise, $|y_3 + y_{-3} - y_1 - y_{-1} - 2(\lambda_3 - \lambda_1)| < \epsilon_3$ implies

$$y_3 + y_{-3} - y_1 - y_{-1} \in \{0, 1, 2, 3\}. \tag{3.5}$$

If $y_2 + y_{-2} - y_1 - y_{-1} = 0$, then $2(\lambda_2 - \lambda_1) < \epsilon_2$ and so $0 \le 2(\lambda_3 - \lambda_1) < \epsilon_2 a$. Hence $y_3 + y_{-3} - y_1 - y_{-1} < \epsilon_2 a + \epsilon_3 = a - 3 \approx 0.34635$, and so $y_3 + y_{-3} - y_1 - y_{-1} = 0$. On the other hand, suppose $y_2 + y_{-2} - y_1 - y_{-1} = 1$. Then $2(\lambda_2 - \lambda_1) > 1 - \epsilon_2$ and so $2(\lambda_3 - \lambda_1) > (1 - \epsilon_2)a = a - \epsilon_2 a$. Hence $y_3 + y_{-3} - y_1 - y_{-1} > a - \epsilon_2 a - \epsilon_3 = 3$, a contradiction to (3.5).

It follows that $y_2 + y_{-2} - y_1 - y_{-1} = y_3 + y_{-3} - y_1 - y_{-1} = 0$. □

To summarize, we have the following reworking of Proposition 17 in [1], but with explicit β_0.

Theorem 3.4. *For each $s \ge 5$, there are no tight $2s$-designs with $\beta > \beta_0(s)$.*

Proof: Suppose $x_{-\lfloor \frac{s}{2} \rfloor} < \cdots < x_{\lfloor \frac{s}{2} \rfloor}$ are the intersection numbers of a tight $2s$-design with $\beta > \beta_0(s)$. By Proposition 2.2, we may assume $t \ge 2$. Then, since $\xi_{-i} = -\xi_i$ and $\lambda_{-i} = \lambda_i$, we have $x_i + x_{-i} - x_j - x_{-j} = y_i + y_{-i} - y_j - y_{-j}$, and this implies that $y_i + y_{-i} - y_j - y_{-j}$ is an integer for each $i, j \in \{(0), 1, 2, \ldots, \lfloor \frac{s}{2} \rfloor\}$. By Propositions 3.1 (iii), 3.2 (iii) and 3.3, these integers must vanish. Specifically,

<u>*Case 1:*</u> s is odd and ≥ 5 implies $y_i + y_{-i} - y_{i-1} - y_{-(i-1)} = 0$ for $1 \le i \le \lfloor \frac{s}{2} \rfloor$.
<u>*Case 2:*</u> s is even and ≥ 8 implies $y_i + y_{-i} - y_{i-1} - y_{-(i-1)} = 0$ for $2 \le i \le \lfloor \frac{s}{2} \rfloor$.
<u>*Case 3:*</u> $s = 6$ implies $y_2 + y_{-2} - y_1 - y_{-1} = y_3 + y_{-3} - y_1 - y_{-1} = 0$.

In each case, the x_i are symmetric about their arithmetic mean $\overline{\alpha}$. By Proposition 2 in [1], this implies $v = 2k + 1$. Proposition 2.3 says this is impossible, and the proof is therefore complete. □

3.2 Searching over small β

We now turn to small values of β, for which the problem becomes finite.

Algorithm 3.5. To exclude tight $2s$-designs with $\beta \leq \beta_0$, we may implement the following steps.

1. Compute β_0 from the B_i, C_i as in the previous section.

2. By Propositions 2.2 and 2.3, we may restrict attention to $t > 2$. Since $\alpha = \beta^2/(1-\frac{1}{t})^2$, it follows that $\alpha < 4\beta_0^2$. Now, since $\alpha = \left(s\overline{\alpha} + \binom{s}{2}\right)/s$ and $s\overline{\alpha}$ is an integer, we have $\alpha \in \frac{1}{s}\mathbb{Z}$. This gives an explicit finite number of admissible α, as Bannai observed in [1].

3. Note that, under the assumption of a tight design, the expression

$$\binom{s}{2}\alpha\left(\alpha + \frac{2\alpha t - \alpha + 2}{\alpha t^2 + t + 1}\right) \tag{3.6}$$

   is an integer. This is because Proposition 5 in [1] asserts that the coefficient of x^{s-2} in the monic polynomial $s!\Psi_s(x)/\binom{v-s}{s}$ is

$$\binom{s}{2}\alpha\left(\alpha + \frac{2\alpha t - \alpha + 2}{\alpha t^2 + t + 1}\right) + \binom{s}{3}\left(3\alpha + \frac{3s-1}{4}\right),$$

   and the latter term is always an integer.

4. Fix α as in Step 2. Put $n = k - s = \alpha t$ and define

$$g_\alpha(n) := \binom{s}{2}\alpha^2\left(1 + \frac{2n - \alpha + 2}{n^2 + n + \alpha}\right)$$

   as in (3.6). As $t > 2$, we may take a lower bound $n_{\min}(\alpha) = \max\{s, \lfloor 2\alpha \rfloor + 1\}$.

5. Since $g'_\alpha(n) < 0$ for all $n \geq n_{\min}(\alpha)$, it suffices to loop on integers n from $n_{\min}(\alpha)$ until $n_{\max}(\alpha)$, where $g_\alpha(n_{\max}(\alpha)) \leq \lfloor \binom{s}{2}\alpha^2 \rfloor + 1$. Any pairs (k, v) which give integral $g_\alpha(n)$ are obtained by $k = n + s$ and $v = \frac{n^2+n}{\alpha} + 2s - 1$.

6. In principle, at this point the zeros of Ψ_s for these pairs (k, v) can be analyzed. However, in practice we found it sufficient in all cases to merely see that $\lambda = \binom{v}{s}\binom{k}{2s}/\binom{v}{2s}$ was never even an integer.

We wrote a C program that implements Algorithm 3.5 for a given s and β_0, but with an important optimization. For n near $n_{\max}(\alpha)$, $|g'_\alpha(n)|$ is very small so it would be inefficient to loop over n in this region. Therefore, the program loops over integer values of $g_\alpha(n)$ from $\lceil g_\alpha(n_{\max}(\alpha)) \rceil$ and checks the integrality of the corresponding n until the derivative becomes larger than a certain threshold (in absolute value), at which point it begins looping over n to a much smaller $n_{\max}$. The program is available by contacting the authors.

Our calculations of β_0 in step 1 are displayed in Appendix B. We can report that the method succeeds for $5 \le s \le 9$, and probably much higher s. We have chosen to avoid continued searches for $s > 9$ until new ideas are obtained. In particular, it would be interesting if $s \ge s_0$ could be excluded for nontrivial tight $2s$-designs.

Theorem 3.6. *For each $5 \le s \le 9$, there are no nontrivial tight $2s$-designs.*

4 The case $s = 4$

The same analytic approach that is successful for $s \ge 5$ fails when $s = 4$. We can only guarantee that

$$0 \le 2(\lambda_2 - \lambda_1) < \frac{\xi_2^2 - \xi_1^2}{3} = \sqrt{8/3} \approx 1.63299$$

when $t \ge 2$, and so there does not exist β_0 such that $|y_2 + y_{-2} - y_1 - y_{-1}| < 1$ for all $\beta > \beta_0$.

However, it is possible to bound $|y_2 + y_{-2} - y_1 - y_{-1}|$ away from 2. Let

$$\epsilon_1 = \frac{1}{2}\left(2 - \sqrt{8/3}\right)\left(1 + \frac{C_2 D_1}{C_1 D_2}\right)^{-1} \quad \text{and} \quad \beta_\star(4) = \widehat{\beta}(1, \epsilon_1).$$

Then the existence of a tight 8-design with $\beta > \beta_\star(4)$ and $t \ge 2$ implies $y_2 + y_{-2} - y_1 - y_{-1} = 1$ and $2(\lambda_2 - \lambda_1) \approx 1$, for which

$$t = \tfrac{v-7}{k-3} \approx \frac{2}{1 - \sqrt[4]{3/8}} \approx 9.1971905725.$$

We are able to obtain more precise conditions in the following result.

Proposition 4.1. *If there exists a nontrivial tight 8-design with parameters v and k, then $k > 25{,}000$ and $f(k, v) = 0$, where $f(k, v)$ is as in Appendix C.*

Proof: We first used Algorithm 3.5 to find that there are no nontrivial tight 8-designs with $\beta \le \beta_\star(4)$. Thus, any tight 8-design with $t \ge 2$ must have $x_2 + x_{-2} - x_1 - x_{-1} = y_2 + y_{-2} - y_1 - y_{-1} = 1$. Consider the monic and root-centered polynomial

$$F(x) = 24\Psi_4(x + \overline{\alpha})/\binom{v-4}{4} = x^4 + p_1 x^3 + p_2 x^2 + p_3 x + p_4.$$

By Equation (15) in [10], we have

$$\begin{aligned}
p_2 &= -\frac{5}{2} - \frac{6(k-3)(k-4)(v-k-3)(v-k-4)}{(v-6)(v-7)^2}, \\
p_3 &= \frac{-4(k-3)(k-4)(v-k-3)(v-k-4)(v-2k+1)(v-2k-1)}{(v-5)(v-6)(v-7)^3}, \\
p_4 &= \frac{9}{16} + \frac{3}{2} \cdot \frac{(k-3)(k-4)(v-k-3)(v-k-4)g(k,v)}{(v-4)(v-5)(v-6)(v-7)^4},
\end{aligned} \tag{4.1}$$

where $g(k,v)$ is as in Appendix C. Assuming $x_2+x_{-2}-x_1-x_{-1}=1$, the roots of $F(x)$ must be $r_1-1/4, r_2+1/4, -r_1-1/4, -r_2+1/4$ where $r_1-1/4=x_1-\overline{\alpha}$ and $r_2+1/4=x_2-\overline{\alpha}$ (note that $r_1,r_2\in\frac{1}{4}\mathbb{Z}$). Expanding,

$$x^4+p_2x^2+p_3x+p_4=x^4+\left(-\tfrac{1}{8}-r_1^2-r_2^2\right)x^2+\left(\tfrac{r_1^2}{2}-\tfrac{r_2^2}{2}\right)x+\left(r_1^2-\tfrac{1}{16}\right)\left(r_2^2-\tfrac{1}{16}\right),$$

which yields

$$p_4=\left(\frac{p_2}{2}+\frac{1}{8}\right)^2-p_3^2. \tag{4.2}$$

Substituting (4.1) into (4.2) results in the equation $f(k,v)=0$. An easy computer search shows that there are no integer solutions v to $f(k,v)=0$ for $9\le k\le 25{,}000$. □

By reducing $f(k,v)$ modulo some primes, one may obtain infinite classes of both k and v which admit no soultions. Some more (very easy) computing is required here.

Corollary 4.2. *There is no nontrivial tight* 8-(v,k,λ) *design with parameters in any of the following congruence classes:*

$$\begin{array}{ll} & v\equiv 3 \pmod 7\\ k\equiv 12 \pmod{13} & v\equiv 3 \pmod{11}\\ k\equiv 5,9,11,12 \pmod{17} & v\equiv 1,3 \pmod{13}\\ & v\equiv 0,1,8,12,13,14 \pmod{17}\end{array}$$

Despite these strict conditions on hypothetical tight 8-designs, it remains open whether there are a finite number of nontrivial such designs.

To loosely summarize our work, we have shown that any unknown tight $2s$-design with $s>1$, if it exists, must have

- large s and small β, or
- $s=4$ with very large k and v satisfying strict conditions.

Appendices

A Hermite polynomials $H_s(x)$ and their zeros, $1\le s\le 9$

s	$H_s(x)$
1	x
2	x^2-1
3	x^3-3x
4	x^4-6x^2+3
5	x^5-10x^3+15x
6	$x^6-15x^4+45x^2-15$
7	$x^7-21x^5+105x^3-105x$
8	$x^8-21x^6+210x^4-420x^2+105$
9	$x^9-36x^7+378x^5-1260x^3+945x$

Note: In the following table, the values of $D_i = |H_{s-1}(\xi_i)|$ are rounded down.

s	i	ξ_i for $H_s(x)$	$D_i \geq$
1	0	0	1
2	1	1	1
3	0	0	1
	1	$\sqrt{3}$	2
4	1	$\sqrt{3-\sqrt{6}} = 0.7420$	1.817
	2	$\sqrt{3+\sqrt{6}} = 2.3344$	5.718
5	0	0	3
	1	$\sqrt{5-\sqrt{10}} = 1.3556$	4.649
	2	$\sqrt{5+\sqrt{10}} = 2.8570$	20.64
6	1	0.61670659019	6.994
	2	1.88917587775	15.02
	3	3.32425743355	88.46

s	i	ξ_i	$D_i \geq$
7	0	0	15
	1	1.1544	20.69
	2	2.3668	57.82
	3	3.7504	433.1
8	1	0.5391	41.09
	2	1.6365	73.30
	3	2.8025	255.7
	4	4.1445	2365
9	0	0	105
	1	1.0233	135.4
	2	2.0768	299.5
	3	3.2054	1267
	4	4.5127	14159

B Constants B_i and C_i obtained from Algorithm 2.8 and values of $\beta_0(s)$

Notes: For convenience, B_i was chosen independently of i and C_i was taken with $C_i = C_{-i}$.

s	B_i	C_i, $i = (0), 1, \ldots, \lfloor \frac{s}{2} \rfloor$	$\beta_0(s)$	$\beta_\star(s)$
4	10	$2, 14$		19.35
5	10	$1, 12, 88$	33.76	
6	100	$11, 63, 558$	156.96	
7	10	$6, 93, 458, 4649$	86.55	
8	100	$100, 501, 2561, 30779$	106.77	
9	100	$9, 773, 3186, 17732, 247789$	146.37	

C The $f(k,v)$ and $g(k,v)$ used in Proposition 4.1

$f(k,v) = -3408102864+1506333312k^2+974873344k^4-488998144k^6+62323584k^8-3309568k^{10}+65536k^{12}+9310949028v-1506333312kv-4733985888k^2v-1949746688k^3v-1015706784k^4v+1466994432k^5v+511604992k^6v-249294336k^7v-49810560k^8v+16547840k^9v+1744896k^{10}v-393216k^{11}v-16384k^{12}v-11097146016v^2+4733985888kv^2+6922441360k^2v^2+2031413568k^3v^2-1428764528k^4v^2-1534814976k^5v^2+209662720k^6v^2+199242240k^7v^2-21567744k^8v^2-8724480k^9v^2+786432k^{10}v^2+98304k^{11}v^2+7281931941v^3-5947568016kv^3-4944873072k^2v^3+412538336k^3v^3+1856597696k^4v^3+243542016k^5v^3-293538048k^6v^3-13016064k^7v^3+17194752k^8v^3-327680k^9v^3-253952k^{10}v^3-2755473732v^4+3929166288kv^4+1497511456k^2v^4-1155170432k^3v^4-582955856k^4v^4+183266304k^5v^4+58253568k^6v^4-16432128k^7v^4-1102464k^8v^4+368640k^9v^4+544096980v^5-1459281552kv^5+28759472k^2v^5+469164960k^3v^5-7038496k^4v^5-59703552k^5v^5+6536960k^6v^5+2050560k^7v^5-328320k^8v^5-$

$18769932v^6 + 293023248kv^6 - 127930016k^2v^6 - 58917568k^3v^6 + 27050224k^4v^6 + 1258752k^5v^6 - 1642240k^6v^6 + 182784k^7v^6 - 14780538v^7 - 24513072kv^7 + 27560816k^2v^7 - 2875616k^3v^7 - 2296192k^4v^7 + 698880k^5v^7 - 61184k^6v^7 + 2961396v^8 - 764688kv^8 - 1582560k^2v^8 + 772608k^3v^8 - 143664k^4v^8 + 10752k^5v^8 - 191952v^9 + 203472kv^9 - 52816k^2v^9 + 7520k^3v^9 - 640k^4v^9 + 972v^{10} - 2352kv^{10} + 336k^2v^{10} + 45v^{11}$.

$g(k,v) = 2k^4v - 26k^4 - 4k^3v^2 + 52k^3v + 2k^2v^3 - 20k^2v^2 - 120k^2v + 258k^2 - 6kv^3 + 120kv^2 - 258kv + v^4 - 23v^3 + 123v^2 - 433v + 764$.

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